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The Properties of Time and Phase Variances in the Presence of Power Law Noise for Various Systems

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Abstract— **This paper discusses the behavior of sample and Allan variances of the time and phase error in the presence of power law noise for a variety of systems. These include various digital, communications, signal processing, radar, and tracking and ranging systems, and time transfer systems. A theory is presented which shows that the sample variance can converge in the presence power law noise for many systems by incorporating an explicit system phase response function $H_s(f)$ into the spectral definition of the variance. $H_s(f)$ is shown to fall into four general classes for the purposes of discussing variance behavior. These classes are: digital sampling, delay, delay with averaging, and PLL. A detailed discussion of the properties of these variances for the above systems, categorized by these four classes of $H_s(f)$, is the heart of the paper.**

I. Introduction

The purpose of this paper is to discuss the behavior of variances of the phase error $\phi(t)$ and the time error $x(t)$ ($x(t)=\phi(t)/\omega_o$, where ω_o is the nominal angular carrier frequency) for various types of systems. Especial attention is given to their properties in the presence of random negative power law noise given by the power spectral density (PSD)

$$S_x(f)\ (\text{or}\ S_\phi(f)) \propto f^{\beta} \tag{1}$$

where $\beta<0$. The systems we will consider include those in the digital, communications, signal processing, radar, ranging, and time transfer areas.

This work is a further application of that presented in refernces [1,2]. It should be noted there are some differences in notation from these references. Among these are the use of the exponent notation in (1) in order to be consistent with that in IEEE Standard 1139 [3]. Other differences will be pointed out as they occur. Since x and ϕ are related by the proportionality constant ω_o, everything we say about one variable in this paper will apply to the other variable through a simple renormalization. For simplicity, we will therefore discuss the properties of either x or ϕ as appropriate or convenient and leave it to the reader to make any transformations as necessary.

The approach for defining variances x and ϕ in this paper is different from that used in prior works of other authors [4-13], It is instructive to discuss these differences before we proceed to the heart of the paper. Two approaches for defining variances of x and ϕ have been used in these prior works. In one such approach, the variance is defined spectrally as [4-6]

$$\sigma_{x\#}^2 = \int_0^{f_h} S_x(f) K_{x\#}(f)\,df \tag{2}$$

In (2), the system phase response is characterized spectrally by a simple high-frequency cut-off f_h and the variance of x is characterized by a spectral kernel $K_{x\#}(f)$. (Note that x# is a place-holder for x1 and x2 to be defined later). The problem with this definition is that the convergence properties of the spectral integral (2) for $\beta<0$ is completely determined by the highpass properties of $K_{x\#}(f)$. For various forms of the Allan variance, this is not an issue because the kernels for these variances have a well-known f^4 dependence for small f [4-6]. Thus (2) converges for Allan variances when $\beta\geq-4$, which is the most negative exponent of $S_x(f)$ encountered in practice. However, many electrical engineering communities other than the precise time and frequency community utilize the standard variance or sample variance of x or ϕ as defined in stationary statistics rather than the Allan variance [7-10]. We will show later in this paper that the kernel for sample variance is proportional to f^2 for small f, which is only sufficient to guarantee the convergence of (2) for $\beta\geq-2$ without the addition of other highpass filtering elements. It is well known that the kernel for the standard variance is unity, so this variance does not converge for $\beta<0$ unless other highpass elements are present.

This convergence problem has been solved in a second prior approach. In this approach, one characterizes the fluctuations of x by using a simple bandpass variance given by

$$\sigma_{x-bp}^2 = \int_{f_l}^{f_h} S_x(f)\,df \tag{3}$$

The variance (3) is often called the jitter [3,11-13], but we will call it a bandpass variance because a heuristic low frequency cut-off f_l is introduced (as well as the high frequency cut-off f_h) to guarantee convergence of the spectral integral for any negative β. However, there is no consistent method for interpreting or determining f_l in the literature [3,11-13], and often there a great deal of confusion about its proper determination.

In the approach we will use in this paper, variances will be defined in spectral form by

$$\sigma_{x\#}^2 = \int_0^{\infty} S_x(f)\,|H_s(f)|^2 K_{x\#}(f)\,df \qquad (4)$$

In (4), we utilize a phase response function $H_s(f)$ determined by explicitly examining filtering properties of the system under consideration. $H_s(f)$ is the single-sided (double sideband) Fourier frequency transform of $h_s(t)$ the time-domain phase response of the system defined, for uniform samples of x at the output of the system, by

$$x_n = \int_{-\infty}^{\infty} dt\, x(t) h_s(n\tau - t) \qquad (5)$$

In the above, τ is the uniform sampling interval that is some multiple of the clock or oscillator period, n is the sample index, and x(t) is the continuous time error from the clock or oscillator driving the system which is characterized by the PSD $S_x(f)$. Figure 1 is a graphical model of the system.

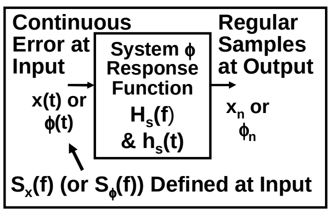

Figure 1. Model of System.

In this paper, we will show that $H_s(f)$ for many types of systems has highpass as well as lowpass properties that aid in the convergence of (4) for $\beta<0$. In these systems, the sample variance is convergent for $\beta \geq -4$ just like the Allan variance. We will also show that $H_s(f)$ falls into four general classes that cover the above mentioned systems for the purposes of characterizing the behavior of variances of x and ϕ in the presence of power law noise. These classes are: digital sampling, delay, delay with averaging, and PLL. The heart of the paper will be a discussion of the variance properties of the above mentioned systems organized by these classes of $H_s(f)$.

II. Types of Variances

In this paper, we will consider the properties four types of variances of x_n (or ϕ_n). The first two variances characterize the fluctuations of N samples of x_n (or ϕ_n) taken over the observation time T is given by

$$T = N\tau \qquad (6)$$

The first is the sample variance of x_n (or ϕ_n) for these N samples, which is well-known from stationary statistics. It is defined in the time-domain by

$$\sigma_{x1}^2 = \frac{1}{N-1}\sum_{n=1}^{N} \langle (x_n - M_x)^2 \rangle \qquad (7)$$

where

$$M_x = N^{-1}\sum_{n=1}^{N} \bar{y}_n \qquad (8)$$

One can show that the kernel of this variance is given by [1,2]

$$K_{x1}(f) = \frac{N}{N-1}\left[1 - \frac{1}{N^2}\frac{\sin^2(N\pi f\tau)}{\sin^2(\pi f\tau)}\right] \qquad (9)$$

The second is the Allan variance of x_n for these N samples, which we will be define as

$$\begin{aligned}\sigma_{x2}^2 &= \left\langle \frac{\tau^2}{N-1}\sum_{n=1}^{N}\left(\left(x_{n+1} - x_n - \frac{x_N - x_0}{N}\right)^2\right)^2 \right\rangle \\ &= \tau^2\sigma_y^2(N,\tau,\tau)\end{aligned} \qquad (9)$$

where $\sigma_y^2(N,\tau,\tau)$ is the N-sample zero dead time variance of the fractional frequency fluctuations y defined in references [4-6]. One can shown that kernel for this variance can be written as [1,2,4-6]

$$K_{x2}(f) = 4\sin^2(\pi f\tau) K_{x1}(f) \qquad (10)$$

For N=2, this variance becomes the “jitter 2” defined in reference [14]. It should also be noted that σ_{x2}^2 defined (9) differs by a factor of 4 from σ_{xa}^2 in references [1,2].

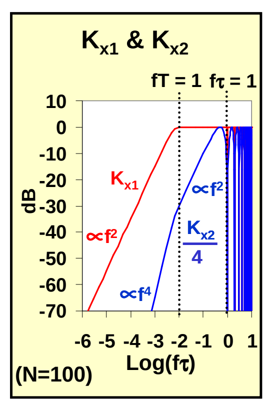

Figure 2. Kernels $K_{x1}(f)$ and $K_{x2}(f)$ vs log(fτ).

The kernels (8) and (10) are plotted in Figure 2. From the equations and the figure, can see that $K_{x1}(f) \propto f^2$, and $K_{x2}(f) \propto f^4$ for $fT \ll 1$. Thus without highpass help from

$H_s(f)$, σ^2_{x1} converges only when $\beta \geq -2$, but σ^2_{x1} converges when $\beta \geq -4$.

For $N >> 1$ but finite T, $K_{x1}(f)$ can be written in the simple form

$$K_{x1}(f) \cong 1 - \text{sinc}^2(\pi f T) \quad [N >> 1, \text{finite T}] \qquad (11)$$

This is very useful in applications where a large number of samples are processed.

When $T \to \infty$, one can see from (11) that $K_{x1}(f) \to 1$, and we obtain the third variance we will consider, the standard variance from stationary statistics given by

$$\lim_{T\to\infty} \sigma^2_{x1} = \sigma^2_{x-std} = \int_0^\infty df \, |H_s(f)|^2 \, S_x(f) \qquad (12)$$

From (12), one can see that σ^2_{x-std} will only converge for $\beta<0$ when $|H_s(f)|^2$ provides sufficient highpass filtering for small f to overcome the f^β behavior of $S_x(f)$.

A power law approximation for K_{x1} that is also very useful for analytic calculations with power law $S_x(f)$ is given by

$$K'_{x1}(fT) = \begin{cases} 3.29(fT)^2 - 2.30(fT)^3 & fT < 1 \\ 1 & fT \geq 1 \end{cases} \qquad (13)$$

From Figure 3, one can see that the difference between K_{x1} and K'_{x1} is very minor, so this is a very good approximation.

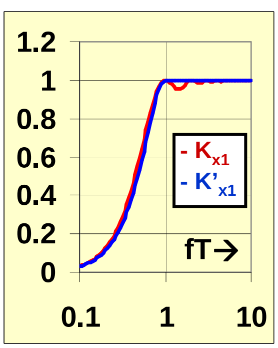

Figure 3. Plot of K_{x1} and K'_{x1}.

The fourth variance we will consider is the bandpass variance defined in (3). We will show that this variance can be obtained as an approximation of either the sample or standard variance for various systems by approximating the filtering properties of $H_s(f)$ and/or $K_{x1}(f)$ with sharp cut-off filters.

III. Variance Properties for Various Systems

Now with above theory as a basis, we can discuss our principal topic, the properties of these variances of time and phase for the above mentioned systems categorized by class of $H_s(f)$.

A. Digital Sampling $H_s(f)$

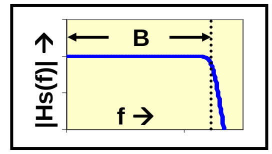

Figure 4. Digital Sampling $H_s(f)$.

The digital sampling response function appears in a variety of digital and signal processing systems. This class of $H_s(f)$ is shown in Figure 4. It is characterized by an anti-aliasing lowpass filter with a baseband real bandwidth of B. B is given, in terms of the sampling interval τ, by

$$\begin{aligned} B &= 0.5/\tau \quad [\text{Nyquist Sampling}] \\ B &< 0.5/\tau \quad [\text{Over-Sampling}] \end{aligned} \qquad (14)$$

where the Nyquist rate just meets the Nyquist criteria for an ideal band limited system without aliasing [15]. For these systems, the number of samples N is generally very large and will be treated as such. The properties of both σ^2_{x1} and σ^2_{x2} for various values of β are listed Table I. The notation in the table is consistent with that in IEEE Standard 1139 [3]. Note that σ^2_{x1} does not converge for $\beta<-2$ because the digital sampling $H_s(f)$ does not contain any highpass filtering, and for $\beta \geq -2$, that it has a very simple form that does not depend on the sampling interval τ in the over-sampled case. (When digital sampling is combined with the other classes to be described, however, σ^2_{x1} can converge for $\beta \geq -4$.) From the table, note also that σ^2_{x2} converges for $\beta \geq -4$, but is a function of the sampling interval τ.

Variance Properties for Digital Sampling T = Data Length, τ = Sampling Interval, B = Anti-Aliasing Filter BW, T >> τ $S_x(f) = g_\beta f^\beta$, $S_y(f) = h_\alpha f^\alpha$, $\beta = \alpha - 2$, $g_\beta = h_\alpha/(2\pi)^2$			
Noise Type	β	σ^2_{x1}/g_β	σ^2_{x2}/g_β [Nyquist: $\tau = 0.5/B$]
White Phase	0	B	$\pi^2\tau^2B^3/3$ [$\pi^2B/12$]
Flicker Phase	-1	0.88+ln(BT)	$\pi^2\tau^2B^2/2$ [$\pi^2/8$]
White Freq	-2	πT	$\pi^2\tau^2B$ [$\pi^2B^{-1}/4 = \pi^2\tau/2$]
Flicker Freq	-3	∞	$\pi^2\tau^2(0.88+\ln(BT))$ [$\pi^2B^{-2}(0.88+\ln(BT))/4$]
Random Run	-4	∞	$\pi^3\tau^2T$ [$\pi^3B^{-2}T/4$]

TABLE I. Variance Properties for Digital Sampling

B. Delay $H_s(f)$

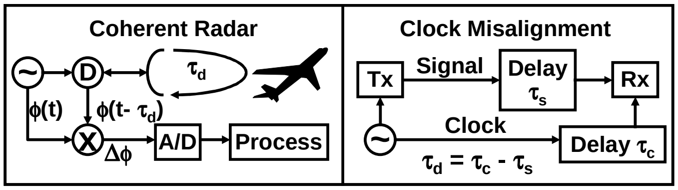

Figure 5. Systems that have a delay $H_s(f)$.

As shown in Figure 5, a delay $H_s(f)$ is encountered in coherent radar and applications where clock or delay misalignment is present, such as synchronous digital systems and in delay line discriminators. This $H_s(f)$ is generated whenever a signal is compared to a version of itself delayed by τ_d. It is characterized by

$$|H_{delay}(f)|^2 = 4\sin^2(\pi f\tau_d) \tag{15}$$

and is shown graphically in Figure 6.

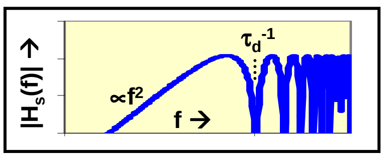

Figure 6. Delay $H_s(f)$.

Note that $|H_{delay}(f)|^2 \propto f^2$ for $f\tau_d << 1$. Thus σ^2_{x1} converges for finite T for β≥-4 because we also have $K_{x1}(f) \propto f^2$ for $fT << 1$. This justifies the heuristic bandpass variance for the phase jitter often used in coherent radar systems given by

$$\sigma^2_{\phi-bp} = \int_{1/T}^{f_h} 4\sin^2(\pi f\tau_d) S_\phi(f)df \tag{16}$$

C. Delay with Averaging $H_s(f)$

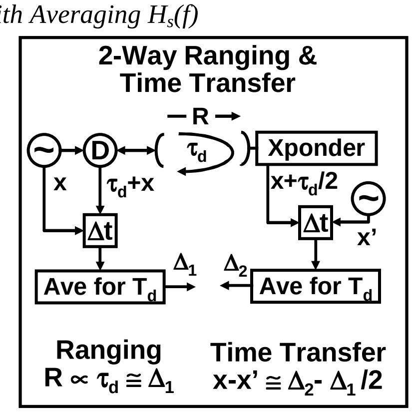

Figure 7. Systems that have delay with averinging $H_s(f)$.

As shown in Figure 7, a delay with averaging $H_s(f)$ is encountered in ranging and 2-way time transfer applications. where a delay response over τ_d is combined with time average over T_d. $H_s(f)$ for these systems is given by

$$|H_{delay/av}(f)|^2 = |H_{delay}(f)|^2 |H_{av}(f)|^2 \tag{17}$$

where

$$|H_{av}(f)|^2 = \sin c^2(\pi f T_d) \tag{18}$$

$|H_{delay/av}(f)|$ is shown in Figure 8.

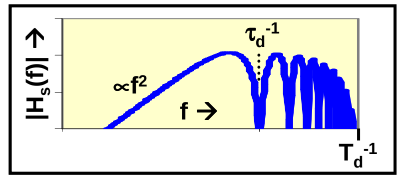

Figure 8. Delay with averaging $H_s(f)$.

For such systems when τ_d=τ and T_d=T, σ^2_{x1} for $H_{delay/av}(f)$ is equivalent to σ^2_{x2} for $H_{delay}(f)$ because of (10). Thus, these are two equivalent pictures of the same systems.

D. PLL $H_s(f)$

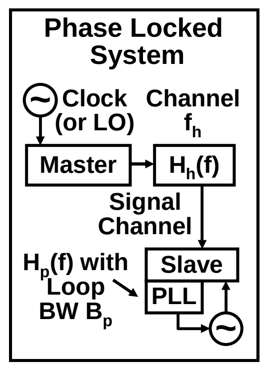

Figure 9. Phase locked system.

Figure 9 shows a typical phase locked system. In such a system, a master clock or local oscillator (LO) is utilized to transmit a signal through a signal channel characterized by the response function $H_h(f)$. A slave or receive system then receives the signal and phase locks a slave clock or LO to the master clock or LO by utilizing a phase locked loop (PLL) tracking on some characteristic of the received signal. The phase locked loop is characterized by a loop response function $H_p(f)$ with a loop bandwidth of B_P. The system response function for this PLL system is thus given by

$$H_{PLL}(f) = [1 - H_p(f)]H_h(f) \tag{19}$$

and is shown in Figure 10.

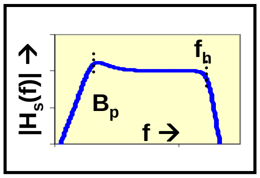

Figure 10. PLL $H_s(f)$.

When $H_{PLL}(f)$ alone is sufficient to guarantee convergence for $S_x(f)$, (16) can be approximated by a bandpass variance

$$\sigma^2_{x-bp} = \int_{B_p}^{f_h} S_x(f) df \quad (17)$$

and the standard variance will converge for negative β. This occurs when β≥-4 for a second order PLL because $|1-H_p(f)|^2 \propto f^4$ for small f, but only occurs when β≥-2 for a first order PLL because $|1-H_p(f)|^2 \propto f^2$ for small f. Thus when β<-2 for a first order PLL, a finite T is required for the convergence of σ^2_{x1} or $\sigma^2_{\phi 1}$. A consequence of this is that the PLL will cycle slip when β<-2 at intervals T approximately given by $\sigma^2_{\phi 1} = 2\pi$ radians for phase-frequency detector and $\sigma^2_{\phi 1} = 1$ radian for sinusoidal phase detector .

IV. CONCLUSIONS

It has been shown that incorporation of a system phase response function $H_s(f)$ into the spectral representation of the sample and standard variances of x or ϕ:

- allows these variances to used in the presence of power law noise for many systems.
- can be used as a means of categorizing the properties of these variances in the presence of power law noise.

Furthermore, the examination of the properties of $H_s(f)$ in the sample variance can be used to

- define the appropriate values of f_h and f_l that appear in the bandpass variance.

It has also been shown that four general classes $H_s(f)$ cover a wide range of systems. These are the: digital sampling, delay, delay with averaging, and PLL $H_s(f)$.

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